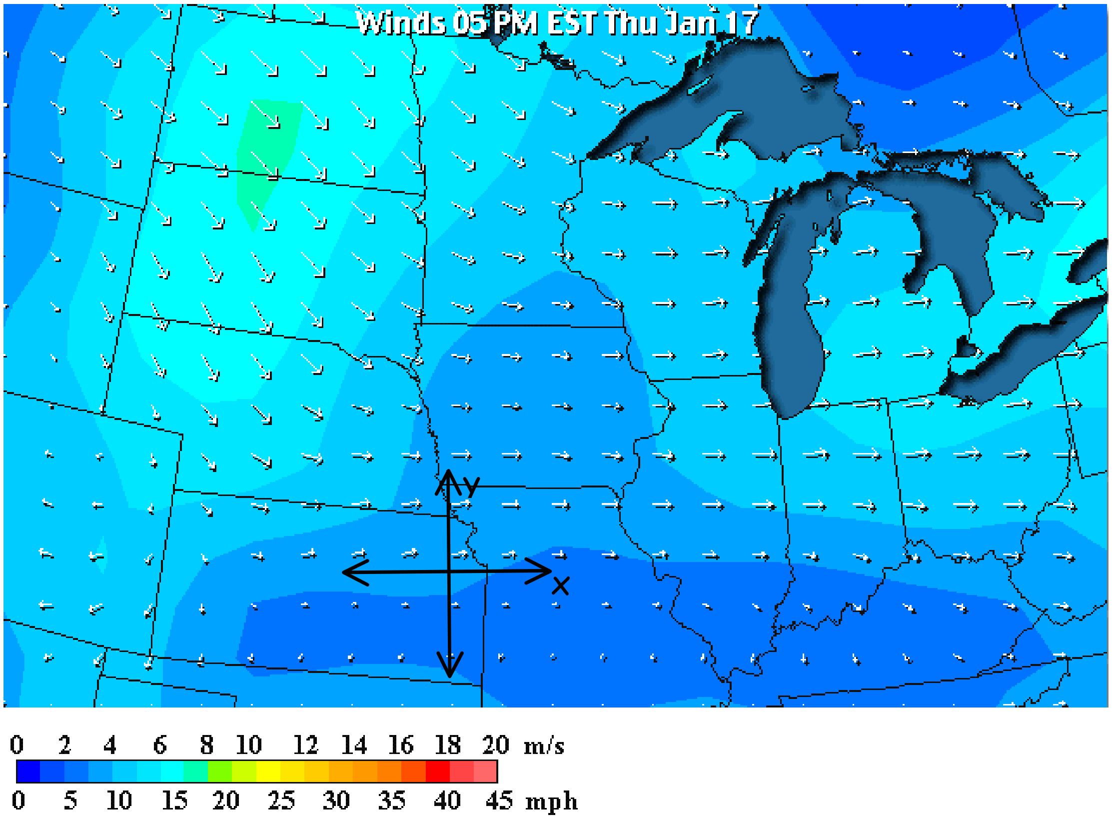
Vector Fields

Base vectors give us a convenient way to express **vector fields**!

You will recall that a **vector field** is a vector quantity that is a **function** of other scalar values. In this class, we will study vector fields that are a function of **position** (e.g., $\mathbf{A}(x,y,z)$).

We earlier considered an **example** of a vector field of this type: the wind **velocity** $\mathbf{v}(x,y)$ across the upper Midwest.



When we express a vector field using orthonormal **base vectors**, the **scalar component** of each direction is a **scalar field**—a scalar function of position!

In other words, a **vector field** can have the form:

$$\mathbf{A}(x,y,z) = A_x(x,y,z)\,\hat{a}_x + A_y(x,y,z)\,\hat{a}_y + A_z(x,y,z)\,\hat{a}_z$$

We therefore can express a **vector** field $\mathbf{A}(x,y,z)$ in terms of **3 scalar** fields: $A_x(x,y,z)$, $A_y(x,y,z)$, and $A_z(x,y,z)$, which express each of the 3 scalar **components** as a **function** of position (x,y,z).

For example, we might encounter this **vector field**:

$$\mathbf{A}(x,y,z) = (x^2 + y^2)\,\hat{a}_x + \frac{xz}{y}\,\hat{a}_y + (3-y)\,\hat{a}_z$$

In this case it is evident that:

$$A_x(x,y,z) = (x^2 + y^2)$$

$$A_y(x,y,z) = \frac{xz}{y}$$

$$A_z(x,y,z) = (3-y)$$

The vector algebraic rules that we discussed in previous handouts are just as **valid** for vector **fields** and scalar **field** components as they are for **discrete** vectors and **discrete** scalar components.

For example, consider these two **vector fields**, expressed in terms of orthonormal **base vectors** $\hat{a}_x, \hat{a}_y, \hat{a}_z$:

$$\mathbf{A}(x,y,z) = y^2\,\hat{a}_x + (x-z)\,\hat{a}_y + \frac{y}{z}\,\hat{a}_z$$

$$\mathbf{B}(x,y,z) = (x+2)\,\hat{a}_x + z\,\hat{a}_y + xyz\,\hat{a}_z$$

The **dot product** of these two vector fields is a **scalar field**:

$$\begin{aligned}\mathbf{A}(x,y,z)\cdot\mathbf{B}(x,y,z) &= A_x B_x + A_y B_y + A_z B_z \\ &= y^2(x+2) + (xz - z^2) + xy^2\end{aligned}$$

Likewise, the **sum** of these two vector fields is a **vector field**:

$$\begin{aligned}\mathbf{A}(x,y,z)+\mathbf{B}(x,y,z) &= (A_x + B_x)\,\hat{a}_x + (A_y + B_y)\,\hat{a}_y + (A_z + B_z)\,\hat{a}_z \\ &= (y^2 + x + 2)\,\hat{a}_x + x\,\hat{a}_y + \frac{y(xz^2+1)}{z}\,\hat{a}_z\end{aligned}$$

Note the example vector fields we have shown here are a function of **spatial** coordinates **only**. In other words, the vector field is **constant** with respect to **time**—the discrete vector quantity at any and every point in space **never changes** its magnitude or direction.

However, we find that many (if not most) vector fields found in nature **do** change with respect to both spatial position **and** time.

Thus, we often discover that vector fields must be written as variables of three spatial coordinates, as well as a **time** variable t!

For example:

$$\mathbf{A}(x,y,z,t) = (x^2 + y^2)t\ \hat{a}_x + \frac{xz}{y}t^2\ \hat{a}_y + (3 - y + 4t)\ \hat{a}_z$$

* A vector field that **changes** with respect to time is known as a **dynamic** vector field.

* A vector field that is **constant** with respect to time is known as a **static** vector field.